## Section 5.4: Indefinite Integrals and the Net Change theorem.

**Objective:** In this lesson, you learn how to

- ☐ Establish indefinite integrals as functions and reformulate the second part of the Fundamental Theorem of Calculus in terms of rate of change and net change.

### I. Indefinite Integrals

$F(x)$

The notation $\int f(x)\,dx = F(x)$ is traditionally used for an **antiderivative** of $f$, that is, $F'(x) = f(x)$ and is called an **indefinite integral**. While a **definite** integral $\int_a^b f(x)\,dx$ is **a number**, an indefinite integral $\int f(x)\,dx$ is a **function** (or family of functions).

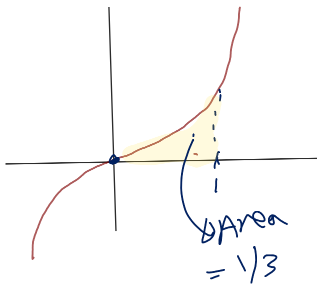
The effectiveness of the Fundamental Theorem depends on having a supply of antiderivatives of functions:

Table of Indefinite Integrals

| | |
|---|---|
| $\int k\,dx = kx + C$ | $\int [c_1 f(x) \pm c_2 g(x)]\,dx = c_1 \int f(x)\,dx \pm c_2 \int g(x)\,dx$ |
| $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$ | $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$ |
| $\int e^x\,dx = e^x + C$ | $\int b^x\,dx = \frac{b^x}{\ln b} + C$ |
| $\int \sin x\,dx = -\cos x + C$ | $\int \cos x\,dx = \sin x + C$ |
| $\int \sec^2 x\,dx = \tan x + C$ | $\int \csc^2 x\,dx = -\cot x + C$ |
| $\int \sec x \tan x\,dx = \sec x + C$ | $\int \csc x \cot x\,dx = -\csc x + C$ |
| $\int \frac{1}{1+x^2}\,dx = \arctan x + C$ $= \tan^{-1} x + C$ | $\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$ $= \sin^{-1} x + C$ |

Comparison between Indefinite and definite integrals:

| | **Indefinite** | **Definite** |
|---|---|---|
| **Notation** | $\int f(x)\,dx$ | $\int_a^b f(x)\,dx$ |
| **Meaning** | Antiderivative | Signed area |
| **Result** | A function of $x$ | A number |
| **Constant** | $C$ | No $C$ |




**Example 1:** Find the following

a. $\int x^2 = \frac{1}{3}x^3 + C$

$\int_0^1 x^2\,dx = \frac{1}{3}x^3 \Big|_0^1$

$= \frac{1}{3}\left[1^3 - 0^3\right] = 1/3$

b. $\int \frac{\sin\theta}{\cos^2\theta}\,d\theta =$

$\int \frac{\sin\theta}{\cos\theta}\,\frac{1}{\cos\theta}\,d\theta$

$= \int \tan\theta\,\sec\theta\,d\theta$

$= \sec\theta + C$

$\int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\,d\theta = \sec\theta \Big|_0^{\pi/4}$

$= \sec\frac{\pi}{4} - \sec 0$

$= \frac{\sqrt{2}}{1} - 1$

Recall

$\tan x = \frac{\sin x}{\cos x}$

$\sec x = \frac{1}{\cos x}$

$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$

$\csc x = \frac{1}{\sin x}$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\sec \pi/4 = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\cos 0 = 1$

$\sec 0 = 1$

$$\int (\sqrt{y}-y)y^{-2}\,dy = \int \sqrt{y}\cdot y^{-2} - y\cdot y^{-2}\,dy = \int y^{1/2}y^{-2} - y y^{-2}\,dy = \int y^{\frac{1}{2}+-2} - y^{1-2}\,dy$$
$$= \int y^{-3/2} - y^{-1}\,dy$$

c. $\displaystyle\int \frac{\sqrt{y}-y}{y^2}\,dy = \int \frac{\sqrt{y}}{y^2} - \frac{y}{y^2}\,dy$

$$= \int \frac{y^{1/2}}{y^2} - \frac{1}{y}\,dy$$
$$= \int y^{-3/2} - \frac{1}{y}\,dy$$
$$= \frac{y^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} - \ln y + C = -2y^{-1/2} - \ln y + C$$
$$= \frac{-2}{\sqrt{y}} - \ln y + C$$

$\displaystyle\int_1^4 \frac{\sqrt{y}-y}{y^2}\,dy = \frac{-2}{\sqrt{y}} - \ln y\,\Big|_1^4$

$$= \left(\frac{-2}{\sqrt{4}_{\,2}} - \ln 4\right) - \left(\frac{-2}{\sqrt{1}} - \cancel{\ln 1}\right)$$
$$= (-1-\ln 4)+2 = 1-\ln 4$$
$$= 1 - 2\ln 2$$

$$\frac{y^{-\frac{1}{2}}}{-\frac{1}{2}}$$
$$y^{-\frac{1}{2}} \div \frac{-1}{2}$$
$$y^{-1/2} * \frac{-2}{1}$$
$$= -2y^{-1/2}$$

d. $\displaystyle\int t^2(3-4t^5)\,dt$

$$= \int 3t^2 - 4t^7\,dt$$
$$= 3\cdot\frac{t^3}{3} - 4\frac{t^8}{8} + C$$
$$= t^3 - \frac{1}{2}t^8 + C$$

e. $\displaystyle\int 4e^r - \sec^2 r\,dr$

$$= 4\int e^r\,dr - \int \sec^2 r\,dr$$
$$= 4e^r - \tan r + C$$

$$\boxed{\begin{aligned}&\sin^2 x + \cos^2 x = 1\\&\sin^2 x = 1-\cos^2 x\\&\cos^2 x = 1-\sin^2 x\end{aligned}}$$

f. $\displaystyle\int \frac{\cos z}{1-\cos^2 z}\,dz$

$$= \int \frac{\cos z}{\sin^2 z}\,dz = \int \frac{\cos z}{\sin z\,\sin z}\,dz$$

$$= \int \cot z\,\csc z\,dz = -\csc z + C.$$

## II. Applications

Note that $F'(x)$ represents the rate of change of $y = F(x)$ with respect to $x$ and $F(b) - F(a)$ is the change in $y$ when $x$ changes from $a$ to $b$. So FTC2 (Fundamental Theorem of Calculus, Part 2) can be reformulated as follows:

**FTC2 (Net Change Theorem):**

The integral of a rate of change is the **net change**:

$$\int_a^b \underbrace{F'(x)}_{\text{rate of change}} dx = \underbrace{F(b) - F(a)}_{\text{net change}}$$

the heigh of my child this number

the heigh of my child last summer

$F(b)-F(a)$ tells us by how much my child grow.

**Remark:** The **total change** is the integral $\int_a^b |F'(x)|\, dx$

**Example 2:** Let $F(x) = 2 - 2e^{-x} - x$, so that, $F'(x) = 2e^{-x} - 1$. Find the net and the total change in $F(x)$ over $[0, 2]$.

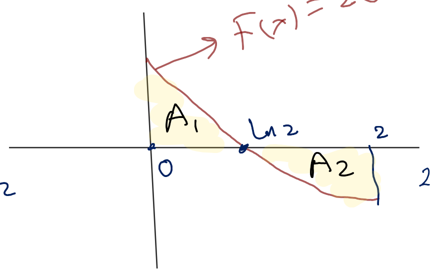
① net change: $\displaystyle\int_0^2 2e^{-x}-1\,dx = 2-2e^{-x}-x\Big|_0^2 = (2-2e^{-2}-2) - (2-2e^{0}-0)$

$= (-2e^{-2}) - (0) = \boxed{-2e^{-2}}$

In terms of net change theorem ⇒ we are computing the difference between the Areas $A_1$ and $A_2$

$$\int_0^2 F'(x)\,dx = A_1 - A_2 = -2e^{-2}$$

$$= \int_0^{\ln 2} 2e^{-x}-1\,dx - \int_{\ln 2}^2 2e^{-x}-1\,dx = -2e^{-2}$$




$2e^{-x}-1=0$

$e^{-x}=\frac{1}{2}$

$\ln e^{-x} = \ln(1/2)$

$-x = -\ln 2$

$x = \ln 2$

② The Total change in $F(x)$ over $[0,2]$

$$\int_0^2 |F'(x)|\,dx = A_1 + A_2' = \int_0^{\ln 2} 2e^{-x}-1\,dx + \int_{\ln 2}^2 -2e^{-x}+1\,dx$$

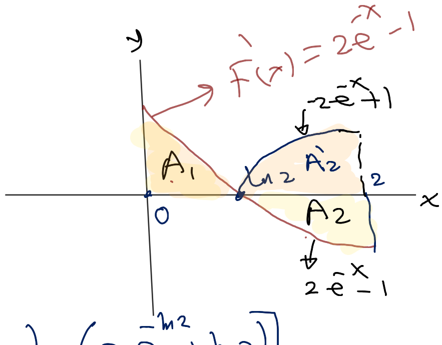
$$= \int_0^{\ln 2} 2e^{-x}-1\,dx - \int_{\ln 2}^2 2e^{-x}-1\,dx$$

$$= -2e^{-x}-x\Big|_0^{\ln 2} + (+2e^{-x}+x)\Big|_{\ln 2}^2$$

$$= \left[(-2e^{-\ln 2}-\ln 2) - (-2-0)\right] + \left[(2e^{-2}+2) - (2e^{-\ln 2}+\ln 2)\right]$$

$$= -1 - \ln 2 + 2 + 2e^{-2} + 2 - 1 - \ln 2$$

$$= \boxed{2 + 2e^{-2} - 2\ln 2}$$




$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$



$\ln 1/2 = \ln 2^{-1} = -\ln 2$

$\ln AB = \ln A + \ln B$

$\ln A/B = \ln A - \ln B$

$\ln A^c = c \ln A$

**Applications in Physics**

The principle can be applied to all of the rates of change in the natural and social sciences. Here are a few examples of this concept:

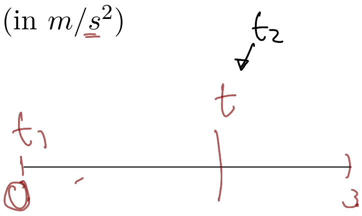
- If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so
$$\int_{t_1}^{t_2} s'(t)\,dt = \int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1),$$
is the net change of position, or displacement, of the particle during the time period from $t_1$ to $t_2$. Similarly,

- The acceleration of the object is $a(t) = v'(t)$, so
$$\int_{t_1}^{t_2} v'(t)\,dt = \int_{t_1}^{t_2} a(t)dt = v(t_2) - v(t_1),$$
is the change in velocity from time $t_1$ to $t_2$.

**Example 3:** A particle is moving a long a line with the acceleration (in $m/s^2$)

$$a(t) = 2t + 3$$

and the initial velocity $\boxed{v(0) = -4\,m/s}$ with $0 \le t \le 3$. Find




a. the velocity at time $t$

$$v(t) - v(t_1) = \int_{t_1}^{t_2} a(r)\,dr \qquad \boxed{t = t_2}$$

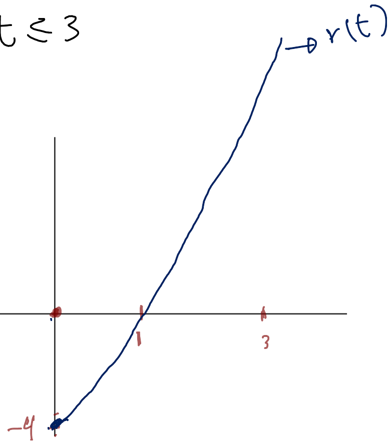
$$v(t) - v(0) = \int_0^t 2r+3\,dr = r^2 + 3r \Big|_0^t = (t^2+3t) - (0+0)$$

$$v(t) - (-4) = t^2 + 3t$$

$$\boxed{v(t) = t^2 + 3t - 4} \qquad 0 \le t \le 3$$

$$= (t-1)(t+4) = 0$$

$$\Rightarrow t = 1 \text{ or } \boxed{t = \cancel{-4}}$$



b. the displacement of the particle during the time period $0 \le t \le 3$.

$$\text{disp} = \int_0^3 v(t)\,dt = \int_0^3 t^2+3t-4\,dt$$

net change

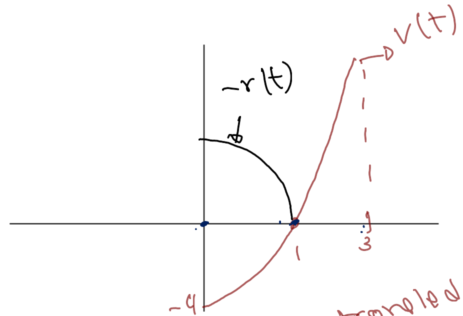
$$= \frac{t^3}{3} + \frac{3}{2}t^2 - 4t \Big|_0^3$$

$$= \left[\frac{3^3}{3} + \frac{3}{2}\cdot(3)^2 - 4(3)\right] - (0)$$

$$= 10.5 \text{ m}$$

total change

c. the distance traveled during the ~~the~~ given time interval.




$$\int_0^3 |s'(t)|\,dt = \int_0^3 |v(t)|\,dt$$

$$= \int_0^1 -t^2-3t+4\,dt + \int_1^3 t^2+3t-4\,dt$$

the distance traveled to the left

$$= \boxed{-\frac{t^3}{3}-\frac{3}{2}t^2+4t\Big|_0^1} + \boxed{\frac{t^3}{3}+\frac{3}{2}t^2-4t\Big|_1^3}$$

→ the distance traveled to right.

$$|v(t)| = \begin{cases} v(t), & v(t)>0 \\ -v(t), & -v(t)<0 \end{cases}$$

$$= \left[\left(\frac{-1}{3} - \frac{3}{2} + 4\right) - (0)\right] + \left[\left(\frac{3^3}{3} + \frac{3}{2}3^2 - 4(3)\right) - \left(\frac{1}{3} + \frac{3}{2} - 4\right)\right]$$

$$= \frac{89}{6} \simeq 14.83$$

The distance traveled during the given time.